

# Rigorous justification of the short-pulse equation

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## Abstract

We prove that the short-pulse equation, which is derived from Maxwell equations with formal asymptotic methods, can be rigorously justified. The justification procedure applies to small-norm solutions of the short-pulse equation. Although the small-norm solutions exist for infinite times and include modulated pulses and their elastic interactions, the error bound for arbitrary initial data can only be controlled over finite time intervals.

## 1 Introduction

Short pulses play an important role in nonlinear optics [2, 6], nonlinear meta-materials [21], and mode-locked lasers [22]. The classical envelope equations such as the nonlinear Schrödinger equation are no longer valid as the pulse width is only few carrier wavelengths, instead of thousands of these. Short-pulse approximations have been derived in this context by using geometric optics [1], nonlocal envelope equations with full dispersion [3, 8], and a regularized nonlinear Schrödinger equation [7]. These models have been rigorously justified similarly to the justification procedure of the classical nonlinear Schrödinger equation [10, 12]. Under the term “rigorous justification”, we understand that the error between solutions of the original and approximated equations is controlled over sufficiently long time intervals.

Another model for short pulses with few cycles on the pulse width was derived by Schäfer & Wayne [19]. We term this model as *the short-pulse equation* and write it in the form,

$$A_{\xi\tau} = A + (A^3)_{\xi\xi}, \quad (1.1)$$

where  $\tau \in \mathbb{R}_+$  is the evolution time,  $\xi \in \mathbb{R}$  is the spatial coordinate, and  $A(\tau, \xi) \in \mathbb{R}$  is the amplitude function. Note that  $A$  is no longer an envelope of harmonic linear waves, compared to the nonlinear Schrödinger approximation.

The short-pulse equation (1.1) represents the class of nonlinear wave equations with low-frequency dispersion, which reduce in the dispersionless limit to the inviscid Burgers equation. Local well-posedness of the short-pulse equation was established in  $H^s(\mathbb{R})$  for  $s > \frac{3}{2}$  [19, 20]. Using a hierarchy

of conserved quantities of the short-pulse equation [4], solutions with small  $H^2$  norm were extended globally for infinite time [16]. On the other hand, solutions with large  $H^2$  norm were proved to blow up in a finite time [13]. The blow-up behavior resembles wave breaking when the amplitude  $A$  remains bounded but the slope steepens up, similar to the self-steeping behavior of the inviscid Burger equation.

Sakovich & Sakovich found that the short-pulse equation (1.1) is integrable by means of the inverse scattering transform [17]. By a coordinate transformation, this equation is reduced to the sine-Gordon equation in characteristic coordinates, which admits exact modulated pulse (breather) solutions [18]. Multi-pulse solutions as well as periodic wave solutions of the short-pulse equation were later found by Matsuno [14, 15].

It is the purpose of this article to justify the applicability of the short-pulse equation (1.1) to dynamics of pulses in the framework of the scalar Maxwell equation,

$$u_{tt} - u_{xx} + u + (u^3)_{xx} = 0, \quad (1.2)$$

where  $t \in \mathbb{R}_+$ ,  $x \in \mathbb{R}$ , and  $u(t, x) \in \mathbb{R}$ . Compared to the full electromagnetic theory, the scalar Maxwell equation (1.2) is only the toy model. It has been used before to construct the breather solutions on a finite spatial scale by means of the spatial dynamics methods [9].

Regarding justifications of the short-pulse equation, Chung *et al.* [5] developed the justification analysis for the linear version of the short-pulse equation by working with oscillatory integrals and roots of the dispersion relations in a more complicated system of Maxwell equations. They also illustrated numerically that the nonlinear version of the short-pulse equation, derived heuristically with a formal renormalization procedure, yields a very good approximation of the modulated pulse solutions in the limit of few cycles on the pulse width.

We shall now develop the nonlinear justification analysis by using the local existence results and apriori energy estimates. For the purpose of justification analysis, it is difficult to work with solutions of the scalar Maxwell equation (1.2) in Sobolev spaces  $H^s(\mathbb{R})$  with higher index  $s > 0$  using the scaled variables of the short-pulse equation (1.1). These norms diverge as  $\epsilon \rightarrow 0$ , the higher is the index, the faster is the divergence. To avoid this difficulty, we shall implement the coordinate transformation from the beginning and work with the error term in the scaled variables. Specifically, we use the transformation of variables,

$$u(t, x) = 2\epsilon U(\tau, \xi), \quad \tau = \epsilon t, \quad \xi = \frac{x - t}{2\epsilon}, \quad (1.3)$$

and rewrite the Maxwell equation (1.2) in the equivalent form,

$$U_{\tau\xi} = U + (U^3)_{\xi\xi} + \epsilon^2 U_{\tau\tau}. \quad (1.4)$$

The short-pulse equation (1.1) appears from the equivalent Maxwell equation (1.4) by neglecting the last term  $\epsilon^2 U_{\tau\tau}$ .

The main result of our justification analysis is the following theorem.

**Theorem 1** Fix  $s > \frac{7}{2}$  and  $T > 0$ . Let  $A \in C([0, T], H^s(\mathbb{R}))$  be a local solution of the short-pulse equation (1.1) such that

$$\sup_{\tau \in [0, T]} \|A(\tau, \cdot)\|_{H^s} + \sup_{\tau \in [0, T]} \|A_\tau(\tau, \cdot)\|_{H^{s-1}} + \sup_{\tau \in [0, T]} \|A_{\tau\tau}(\tau, \cdot)\|_{H^{s-2}} + \sup_{\tau \in [0, T]} \|A_{\tau\tau\tau}(\tau, \cdot)\|_{H^{s-3}} \leq \delta, \quad (1.5)$$

for some  $\delta > 0$ . Assume that there is  $\epsilon > 0$ ,  $U_0 \in H^3(\mathbb{R})$ , and  $V_0 \in H^2(\mathbb{R})$  such that

$$\|U_0 - A(0, \cdot)\|_{H^2} + \|V_0 - A_\tau(0, \cdot)\|_{H^1} \leq \epsilon. \quad (1.6)$$

For a sufficiently small  $\delta > 0$ , there exist  $\epsilon_0 > 0$  and  $C_0 > 0$  such that for all  $\epsilon \in (0, \epsilon_0)$  there exists a unique solution

$$U \in C([0, T], H^2(\mathbb{R})) \cap C^1([0, T], H^1(\mathbb{R})) \cap C^2([0, T], L^2(\mathbb{R})),$$

of the Maxwell equation (1.4) subject to the initial data  $U(0, \cdot) = U_0$ ,  $U_\tau(0, \cdot) = V_0$  satisfying

$$\sup_{\tau \in [0, T]} \|U(\tau, \cdot) - A(\tau, \cdot)\|_{H^2} \leq C_0 \epsilon. \quad (1.7)$$

**Remark 1** Condition (1.5) can be satisfied from constraints on the initial data of the short-pulse equation (1.1), see Corollaries 1, 2, and 3. Loosely speaking, these constraints are satisfied when the first three anti-derivatives of  $A(0, \xi)$  in  $\xi$  are square integrable.

**Remark 2** In terms of the variables of the original Maxwell equation (1.2), we can rewrite the constraints (1.6) and (1.7) in the equivalent form,

$$\left\| u(0, \cdot) - 2\epsilon A\left(0, \frac{\cdot}{2\epsilon}\right) \right\|_{H^2} \leq C\epsilon^{1/2}, \quad \left\| u_t(0, \cdot) + A_\xi\left(0, \frac{\cdot}{2\epsilon}\right) \right\|_{H^1} \leq C\epsilon^{1/2}, \quad (1.8)$$

and

$$\sup_{t \in [0, T/\epsilon]} \left\| u(t, \cdot) - 2\epsilon A\left(\epsilon t, \frac{\cdot - t}{2\epsilon}\right) \right\|_{H^2} \leq C_0 \epsilon^{1/2}. \quad (1.9)$$

for some  $C, C_0 > 0$ . If  $A_0 \in H^2(\mathbb{R})$  and  $\epsilon \rightarrow 0$ , then

$$\left\| \epsilon A_0\left(\frac{\cdot}{2\epsilon}\right) \right\|_{H^2} = \mathcal{O}(\epsilon^{-1/2}), \quad \left\| A'_0\left(\frac{\cdot}{2\epsilon}\right) \right\|_{H^1} = \mathcal{O}(\epsilon^{-1/2}). \quad (1.10)$$

Bounds (1.8), (1.9), and (1.10) show that the error terms between solutions of the short-pulse equation and the scalar Maxwell equation in the original variables are also  $\mathcal{O}(\epsilon)$  smaller than the leading order terms.

**Organization of the paper:** Sections 2 and 3 describe local solutions of the short-pulse and Maxwell equations respectively. Section 4 is devoted to energy estimates for the error term between solutions of the short-pulse and Maxwell equations. Section 5 gives the proof of Theorem 1 by using a continuation argument together with the energy estimates.

**Notations:**  $H^s(\mathbb{R})$  for  $s \geq 0$  denotes the Hilbert–Sobolev space equipped with the norm

$$\|f\|_{H^s} = \left( \int_{\mathbb{R}} (1+k^2)^s |\hat{f}(k)|^2 dk \right)^{1/2},$$

where  $\hat{f}$  is the Fourier transform of  $f$ . We shall intersect these spaces with  $\dot{H}^{-m}$ ,  $m \in \mathbb{N}$ , equipped with the norm

$$\|f\|_{\dot{H}^{-m}} = \left( \int_{\mathbb{R}} k^{-2m} |\hat{f}(k)|^2 dk \right)^{1/2}.$$

If  $f \in \dot{H}^{-m}(\mathbb{R})$ , then the  $m$ -th order anti-derivative of  $f$  is square integrable.

We define the anti-derivative of  $f \in L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$  by

$$\partial_{\xi}^{-1} f := \int_{-\infty}^{\xi} f(\xi') d\xi'.$$

Under the condition  $f \in L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ , the anti-derivative of  $f$  is not only square integrable, but also continuous and decaying to zero as  $|\xi| \rightarrow \infty$  thanks to Sobolev embedding. In particular,  $f$  is the mean-zero function satisfying the constraint  $\int_{-\infty}^{\infty} f(\xi) d\xi = 0$ .

Constant  $C$  stands for a generic  $\epsilon$ -independent positive constant, which may change from one line to another line.

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## 2 Local solutions of the short-pulse equation

We shall start with the local well-posedness of the short-pulse equation (1.1). An improved local existence result is obtained by Stefanov *et al.* [20, Theorem 1]. The following statement will be used in the estimates for the error terms generated by the local solutions of the short-pulse equation.

**Lemma 1** [20] *Fix  $s > \frac{3}{2}$ . For any  $A_0 \in H^s(\mathbb{R})$ , there exists a time  $\tau_0 = \tau_0(\|A_0\|_{H^s}) > 0$  and a unique strong solution of the short-pulse equation (1.1) such that*

$$A \in C([0, \tau_0], H^s(\mathbb{R})) \cap C^1((0, \tau_0], H^{s-1}(\mathbb{R})) \quad (2.1)$$

*and  $A(0, \cdot) = A_0$ . Moreover, the local solution depends continuously on the initial data  $A_0$ .*

We will need some estimates on the higher derivatives of the local solution  $A$  with respect to  $\tau$ . Applying the anti-derivative  $\partial_{\xi}^{-1}$  to locally integrable functions in the distribution sense, we obtain from the short-pulse equation (1.1),

$$A_{\tau} = \partial_{\xi}^{-1} A + (A^3)_{\xi}, \quad (2.2)$$

$$A_{\tau\tau} = \partial_\xi^{-2} A + 3(A^2)_\xi \partial_\xi^{-1} A + 4A^3 + \frac{9}{5}(A^5)_{\xi\xi}, \quad (2.3)$$

$$\begin{aligned} A_{\tau\tau\tau} = & \partial_\xi^{-3} A + \partial_\xi^{-1} A^3 + 18A^2 \partial_\xi^{-1} A + 3(A^2)_\xi \partial_\xi^{-2} A + 6A_\xi (\partial_\xi^{-1} A)^2 \\ & + \frac{27}{2}(A^4)_{\xi\xi} \partial_\xi^{-1} A + \frac{123}{5}(A^5)_\xi + \frac{27}{7}(A^7)_{\xi\xi\xi}, \end{aligned} \quad (2.4)$$

This chain of equations shows that the derivatives of the local solution  $A$  in  $\tau$  can be controlled if the anti-derivatives of  $A$  in  $\xi$  are controlled. The following lemma gives an useful result for this purpose.

**Lemma 2** *Let  $B_0 \in L^2(\mathbb{R})$  and either (a)  $F = G_\xi$  with  $G \in C([0, \tau_0], L^2(\mathbb{R}))$  or (b)  $F \in C^1([0, \tau_0], L^2(\mathbb{R}))$  for some  $\tau_0 > 0$ . The linear inhomogeneous short-pulse equation,*

$$\left. \begin{aligned} B_{\tau\xi} &= B + F, \\ B(0, \cdot) &= B_0, \end{aligned} \right\} \quad (2.5)$$

*admits a unique solution  $B \in C([0, \tau_0], L^2(\mathbb{R}))$ .*

*Proof.* Let  $S(\tau) = e^{\tau \partial_\xi^{-1}} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  denote the fundamental solution operator associated with the linear short-pulse equation  $B_{\tau\xi} = B$ . Using the Fourier transform, we see that the operator  $S(\tau)$  is norm-preserving for any  $\tau \in \mathbb{R}$  in the sense  $\|S(\tau)B_0\|_{L^2} = \|B_0\|_{L^2}$  for any  $B_0 \in L^2(\mathbb{R})$ .

In case (a), we rewrite (2.5) in the integral form,

$$B(\tau, \cdot) = S(\tau)B_0 + \int_0^\tau S(\tau - \tau')G(\tau', \cdot)d\tau'. \quad (2.6)$$

From the norm-preserving property of  $S(\tau)$  and the assumption on  $G$  in (a), we obtain a unique solution  $B \in C([0, \tau_0], L^2(\mathbb{R}))$ .

In case (b), using the decomposition  $B = -F + \tilde{B}$ , we rewrite the initial-value problem (2.5) in the equivalent form,

$$\left. \begin{aligned} \tilde{B}_\tau &= \partial_\xi^{-1} \tilde{B} + F_\tau, \\ \tilde{B}(0, \cdot) &= \tilde{B}_0, \end{aligned} \right\} \quad (2.7)$$

where  $\tilde{B}_0 = B_0 + F(0, \cdot) \in L^2(\mathbb{R})$ . By Duhamel's principle, the initial-value problem (2.7) can be written in the integral form,

$$\tilde{B}(\tau, \cdot) = S(\tau)\tilde{B}_0 + \int_0^\tau S(\tau - \tau')F_\tau(\tau', \cdot)d\tau'. \quad (2.8)$$

From the norm-preserving property of  $S(\tau)$  and the assumption on  $F$  in (b), we obtain a unique solution  $\tilde{B} \in C([0, \tau_0], L^2(\mathbb{R}))$  and hence the assertion of the lemma.  $\square$

We shall now use Lemmas 1 and 2 to control the anti-derivatives of the local solution  $A$  in  $\xi$ .

**Corollary 1** *Fix  $s > \frac{3}{2}$ . If  $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ , then the local solution of Lemma 1 satisfies*

$$\partial_\xi^{-1} A \in C([0, \tau_0], H^{s+1}(\mathbb{R})), \quad A \in C^1([0, \tau_0], H^{s-1}(\mathbb{R})). \quad (2.9)$$

*Proof.* Because  $A \in C([0, \tau_0], H^s(\mathbb{R}))$  from Lemma 1, we only need to prove that  $\partial_\xi^{-1}A \in C([0, \tau_0], L^2(\mathbb{R}))$  in order to show that  $\partial_\xi^{-1}A \in C([0, \tau_0], H^{s+1}(\mathbb{R}))$ . Then,  $A \in C^1([0, \tau_0], H^{s-1}(\mathbb{R}))$  from equation (2.2).

Let us denote  $B^{(1)} := \partial_\xi^{-1}A$ . From equation (2.2), we can see that it satisfies

$$B_{\tau\xi}^{(1)} = B^{(1)} + (A^3)_\xi.$$

Recall that  $H^s(\mathbb{R})$  is a Banach algebra with respect to pointwise multiplication for any  $s > \frac{1}{2}$ . By Lemma 2 in case (a), if  $B_0^{(1)} \in L^2(\mathbb{R})$ , then  $B^{(1)} \in C([0, \tau_0], L^2(\mathbb{R}))$ .  $\square$

**Corollary 2** *Fix  $s > \frac{5}{2}$ . If  $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-2}(\mathbb{R})$ , then the local solution of Lemma 1 satisfies*

$$\partial_\xi^{-2}A \in C([0, \tau_0], H^{s+2}(\mathbb{R})), \quad \partial_\xi^{-1}A \in C^1([0, \tau_0], H^s(\mathbb{R})), \quad A \in C^2([0, \tau_0], H^{s-2}(\mathbb{R})). \quad (2.10)$$

*Proof.* Denote  $B^{(2)} := \partial_\xi^{-2}A$  and compute

$$B_{\xi\tau}^{(2)} = B_\tau^{(1)} = B^{(2)} + A^3.$$

We note that  $F = A^3 \in C^1([0, \tau_0], L^2(\mathbb{R}))$  because of property (2.9). By Lemma 2 in case (b), if  $B_0^{(2)} \in L^2(\mathbb{R})$ , then  $B^{(2)} \in C([0, \tau_0], L^2(\mathbb{R}))$ . Hence  $\partial_\xi^{-2}A \in C([0, \tau_0], H^{s+2}(\mathbb{R}))$  and  $\partial_\xi^{-1}A \in C^1([0, \tau_0], H^s(\mathbb{R}))$ . Then,  $A \in C^2([0, \tau_0], H^{s-2}(\mathbb{R}))$  follows from property (2.9) and equation (2.3).  $\square$

**Corollary 3** *Fix  $s > \frac{7}{2}$ . If  $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-2}(\mathbb{R})$  and  $\partial_\xi^{-3}A_0 + \partial_\xi^{-1}A_0^3 \in L^2(\mathbb{R})$ , then the local solution of Lemma 1 satisfies*

$$A \in C^3([0, \tau_0], H^{s-3}(\mathbb{R})) \quad (2.11)$$

*Proof.* Denote  $B^{(3)} := \partial_\xi^{-3}A + \partial_\xi^{-1}A^3$  and compute

$$B_{\xi\tau}^{(3)} = B^{(3)} + 3A^2\partial_\xi^{-1}A + 9A^4A_\xi.$$

We note that  $F = 3A^2\partial_\xi^{-1}A + 9A^4A_\xi \in C^1([0, \tau_0], L^2(\mathbb{R}))$  because of property (2.10). By Lemma 2 in case (b), if  $B_0^{(3)} \in L^2(\mathbb{R})$ , then  $B^{(3)} \in C([0, \tau_0], L^2(\mathbb{R}))$ . Then,  $A \in C^3([0, \tau_0], H^{s-3}(\mathbb{R}))$  follows from property (2.10) and equation (2.4).  $\square$

Small-norm solutions are known to exist for infinite time of the short-pulse equation. This result was originally proved in  $H^2$  [16, Theorem 1]. Using the blow-up alternative for the short-pulse equation [13, Lemma 2], one can extend this result to any  $s \geq 2$ . To be precise, we have the following result.

**Lemma 3** [13, 16] *Fix  $s \geq 2$ . If  $A_0 \in H^s(\mathbb{R})$  and*

$$\|A'_0\|_{L^2}^2 + \|A''_0\|_{L^2}^2 < \frac{1}{6}, \quad (2.12)$$

*the maximal existence time of the local solution of Lemma 1 extends to infinity. Moreover, there exists  $C > 0$  and a unique solution  $A \in C(\mathbb{R}_+, H^s(\mathbb{R}))$  of the short-pulse equation (1.1) with  $A(0, \cdot) = A_0$  such that  $\|A(\tau, \cdot)\|_{H^s} \leq C$  for all  $\tau \in \mathbb{R}_+$ .*

In the justification analysis, we need small-norm solutions to control linear error terms. However, the justification analysis only holds on finite time intervals in  $\tau$ .

### 3 Local solutions of the scalar Maxwell equation

Local well-posedness of the quasi-linear equations was studied by Kato [11]. To employ his formalism, we shall rewrite the scalar Maxwell equation (1.2) as a system of quasi-linear equations with a symmetric matrix. Because solutions of the short-pulse equation are small solutions of the Maxwell equation in the  $L^\infty$  norm, we can assume that  $\|u\|_{L^\infty} \leq \frac{1}{\sqrt{3}}$  and write

$$u_1 = u_t, \quad u_2 = (1 - 3u^2)^{1/2}u_x, \quad u_3 = u. \quad (3.1)$$

The scalar Maxwell equation (1.2) is equivalent to the system of first-order quasi-linear equations,

$$\frac{\partial}{\partial t} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & -(1 - 3u_3^2)^{1/2} & 0 \\ -(1 - 3u_3^2)^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_3 - \frac{3u_2^2 u_3}{1 - 3u_3^2} \\ -\frac{3u_1 u_2 u_3}{1 - 3u_3^2} \\ u_1 \end{bmatrix}. \quad (3.2)$$

By Theorems II and III in [11], we have the existence of a unique solution of system (3.2) for the vector  $(u_1, u_2, u_3)$  in space  $C([0, t_0], H^s(\mathbb{R})) \cap C^1([0, t_0], H^{s-1}(\mathbb{R}))$  for some  $t_0 > 0$  and  $s > \frac{3}{2}$ . Coming back to the scalar Maxwell equation (1.2), this result is formulated as follows.

**Lemma 4** [11] *Fix  $s > \frac{3}{2}$ . For any  $u_0 \in H^{s+1}(\mathbb{R})$  and  $v_0 \in H^s(\mathbb{R})$  such that  $\|u_0\|_{L^\infty} < \frac{1}{\sqrt{3}}$ , there exists a time  $t_0 = t_0(\|u_0\|_{H^{s+1}} + \|v_0\|_{H^s}) > 0$  and a unique strong solution of the scalar Maxwell equation (1.2) such that*

$$u \in C([0, t_0], H^{s+1}(\mathbb{R})) \cap C^1([0, t_0], H^s(\mathbb{R})) \cap C^2([0, t_0], H^{s-1}(\mathbb{R})), \quad (3.3)$$

*subject to the initial data  $u(0, \cdot) = u_0$  and  $u_t(0, \cdot) = v_0$ . Moreover, the local solution depends continuously on the initial data  $(u_0, v_0)$ .*

Since the existence time  $t_0$  may depend on the initial norm  $\|u_0\|_{H^{s+1}} + \|v_0\|_{H^s}$ , it may be difficult to continue the local solution for infinite time if the norms increase along the local solution. In some cases, blow-up in a finite time is possible in the  $H^{s+1}$  norm for  $u(t, \cdot)$ . Although we are not able to eliminate the blow-up in a finite time, we know from the main result (a-ii) of Yin [23, Theorem 2.3] that the blow-up occurs simultaneously in all norms in  $H^{s+1}$  for any  $s > \frac{3}{2}$ . In other words, we have the following lemma.

**Lemma 5** [23] *The maximal existence time for the local solution in Lemma 4 is independent of  $s > \frac{3}{2}$  in the following sense. If two local solutions of the scalar Maxwell equation (1.2) exist*

$$u \in C([0, t_1], H^{s_1+1}(\mathbb{R})) \cap C^1([0, t_1], H^{s_1}(\mathbb{R})) \cap C^2([0, t_1], H^{s_1-1}(\mathbb{R})), \quad (3.4)$$

*and*

$$u \in C([0, t_2], H^{s_2+1}(\mathbb{R})) \cap C^1([0, t_2], H^{s_2}(\mathbb{R})) \cap C^2([0, t_2], H^{s_2-1}(\mathbb{R})), \quad (3.5)$$

*for the same initial data  $u_0 \in H^{s_1+1}(\mathbb{R}) \cap H^{s_2+1}(\mathbb{R})$  and  $v_0 \in H^{s_1}(\mathbb{R}) \cap H^{s_2}(\mathbb{R})$  with  $s_1, s_2 > \frac{3}{2}$  and  $s_1 \neq s_2$ , then  $t_1 = t_2$ .*

Results of Lemmas 4 and 5 are useful to establish the precise blow-up criterion for the Maxwell equation (1.2), which will allow us to prove the estimates for the approximation in the  $H^2$ -norm and to avoid energy estimates in higher Sobolev spaces.

**Lemma 6** *The local solution in Lemma 4 blows up in a finite time  $t_0 < \infty$  if and only if*

$$\limsup_{t \rightarrow t_0} (\|u(t, \cdot)\|_{L^\infty} + \|u_t(t, \cdot)\|_{L^\infty} + \|u_x(t, \cdot)\|_{L^\infty}) = \infty. \quad (3.6)$$

*Proof.* We only need to prove the necessary condition since if condition (3.6) is satisfied, then the local solution  $u$  blows up in  $H^{s+1}$ -norm for  $s > \frac{3}{2}$ . In order to prove the necessary condition, we proceed by the contradiction. We assume that the solution blows up in the  $H^s$  norm in a finite time  $t_0 < \infty$  but  $M_0, M_1, M_2 < \infty$ , where

$$M_0 = \sup_{t \in [0, t_0]} \|u(t, \cdot)\|_{L^\infty}, \quad M_1 = \sup_{t \in [0, t_0]} \|u_t(t, \cdot)\|_{L^\infty}, \quad M_2 = \sup_{t \in [0, t_0]} \|u_x(t, \cdot)\|_{L^\infty}. \quad (3.7)$$

Because of the independence of the blow-up time from the index  $s$  in Lemma 5, it suffices to consider the simplest  $H^{s+1}$  norm for  $u$  with  $s = 2 > \frac{3}{2}$ .

Let us define the sequence of energies for the scalar Maxwell equation (1.2),

$$E_1(u) = \int_{\mathbb{R}} (u^2 + u_t^2 + u_x^2(1 - 3u^2)) dx, \quad (3.8)$$

$$E_2(u) = \int_{\mathbb{R}} (u_x^2 + u_{tx}^2 + u_{xx}^2(1 - 3u^2)) dx, \quad (3.9)$$

$$E_3(u) = \int_{\mathbb{R}} (u_{xx}^2 + u_{txx}^2 + u_{xxx}^2(1 - 3u^2)) dx. \quad (3.10)$$

As previously, we consider small-norm solutions with  $M_0 < \frac{1}{\sqrt{3}}$ .

Multiplying equation (1.2) by  $u_t$ , we obtain the energy balance equation,

$$\frac{1}{2} \frac{dE_1(u)}{dt} = -3 \int_{\mathbb{R}} uu_t u_x^2 dx, \quad t \in [0, t_0], \quad (3.11)$$

where the decay of  $u, u_t, u_x$  to 0 as  $|x| \rightarrow \infty$  is used. This decay is justified for any local solution of Lemma 4. Under the assumption (3.7), there is  $C(M_0) > 0$  such that

$$\left| \frac{dE_1(u)}{dt} \right| \leq C(M_0) M_0 M_1 E_1(u) \quad \Rightarrow \quad E_1(u) \leq E_1(u_0) e^{C(M_0) M_0 M_1 t}, \quad t \in [0, t_0]. \quad (3.12)$$

Therefore,  $E_1(u)$  cannot blow up in a finite time  $t_0$  if  $M_0, M_1 < \infty$ .

Differentiating equation (1.2) in  $x$  and multiplying the resulting equation by  $u_{tx}$ , we obtain the energy balance equation,

$$\frac{1}{2} \frac{dE_2(u)}{dt} = -3 \int_{\mathbb{R}} uu_t u_{xx}^2 dx - 6 \int_{\mathbb{R}} u_x^3 u_{tx} dx - 12 \int_{\mathbb{R}} uu_x u_{xx} u_{tx} dx, \quad t \in [0, t_0], \quad (3.13)$$



where the decay of  $u_{tx}, u_{xx}$  to 0 as  $|x| \rightarrow \infty$  is used. Again, this decay is justified for any local solution of Lemma 4. Under the assumption (3.7), there is  $C(M_0) > 0$  such that

$$\begin{aligned} \left| \frac{dE_2(u)}{dt} \right| &\leq C(M_0)(M_0M_1 + 2M_2^2 + 4M_0M_2)E_2(u) \\ \Rightarrow E_2(u) &\leq E_2(u_0)e^{C(M_0)(M_0M_1+2M_2^2+4M_0M_2)t}, \quad t \in [0, t_0]. \end{aligned} \quad (3.14)$$

Therefore,  $E_2(u)$  cannot blow up in a finite time  $t_0$  if  $M_0, M_1, M_2 < \infty$ .

We need one more computation for  $E_3(u)$  to obtain a contradiction in the space  $H^3$  for  $u$ . However, because of the integration over  $x \in \mathbb{R}$ , we can not work directly with the local solution  $u$  and need the approximating sequence  $\{u^{(n)}\}_{n \in \mathbb{N}}$  of local solutions in Sobolev space of a higher index  $s = 3 > \frac{3}{2}$ . Applying density arguments and continuous dependence from initial data, we approximate the initial value  $u_0 \in H^3(\mathbb{R})$  and  $v_0 \in H^2(\mathbb{R})$  by functions  $u_0^{(n)} \in H^4(\mathbb{R})$  and  $v_0^{(n)} \in H^3(\mathbb{R})$ , such that  $u_0^{(n)} \rightarrow u_0$  in  $H^3$  and  $v_0^{(n)} \rightarrow v_0$  in  $H^2$  as  $n \rightarrow \infty$ . The approximating sequence  $\{u^{(n)}\}_{n \in \mathbb{N}}$  of local solutions of the Maxwell equation (1.2) is generated by the sequence of the initial data  $\{u_0^{(n)}\}_{n \in \mathbb{N}}$  and  $\{v_0^{(n)}\}_{n \in \mathbb{N}}$ .

Differentiating equation (1.2) twice in  $x$  and multiplying the resulting equation by  $u_{txx}$ , we obtain the energy balance equation,

$$\begin{aligned} \frac{1}{2} \frac{dE_3(u^{(n)})}{dt} &= -3 \int_{\mathbb{R}} u^{(n)} u_t^{(n)} (u_{xxx}^{(n)})^2 dx - 36 \int_{\mathbb{R}} (u_x^{(n)})^2 u_{xx}^{(n)} u_{txx}^{(n)} dx - 18 \int_{\mathbb{R}} u^{(n)} u_x^{(n)} u_{xxx}^{(n)} u_{txx}^{(n)} dx \\ &\quad - 18 \int_{\mathbb{R}} u^{(n)} (u_{xx}^{(n)})^2 u_{txx}^{(n)} dx, \quad t \in [0, t_0], \end{aligned} \quad (3.15)$$

where the decay of  $u_{txx}^{(n)}, u_{xxx}^{(n)}$  to 0 as  $|x| \rightarrow \infty$  is used. This decay is justified for the approximating sequence  $\{u^{(n)}\}_{n \in \mathbb{N}}$  of local solutions of Lemma 4 with  $s = 3$ . Under the assumption (3.7) for the approximating sequence  $\{u^{(n)}\}_{n \in \mathbb{N}}$  rewritten as  $M_0^{(n)}, M_1^{(n)}, M_2^{(n)} < \infty$ , there is  $C(M_0^{(n)}) > 0$  such that

$$\left| \frac{dE_3(u^{(n)})}{dt} \right| \leq C(M_0^{(n)})(M_0^{(n)}M_1^{(n)} + 12(M_2^{(n)})^2 + 9M_0^{(n)}M_2^{(n)} + 9M_0^{(n)}E_2^{1/2}(u^{(n)}))E_3(u^{(n)}), \quad (3.16)$$

where the Gagliardo–Nirenberg inequality is used to estimate the last term of (3.15),

$$\begin{aligned} \left| \int_{\mathbb{R}} u^{(n)} (u_{xx}^{(n)})^2 u_{txx}^{(n)} dx \right| &\leq M_0^{(n)} \|u_{txx}^{(n)}\|_{L^2} \|u_{xx}^{(n)}\|_{L^4}^2 \\ &\leq M_0^{(n)} \|u_{txx}^{(n)}\|_{L^2} \|u_{xxx}^{(n)}\|_{L^2}^{1/2} \|u_{xx}^{(n)}\|_{L^2}^{3/2} \\ &\leq C(M_0^{(n)}) E_2^{1/2}(u^{(n)}) E_3(u^{(n)}). \end{aligned}$$

Since  $E_3(u_0^{(n)}) \rightarrow E_3(u_0)$  as  $n \rightarrow \infty$ , we infer from the continuous dependence of the local solution  $u$  on initial data  $u_0$  that  $E_3(u)$  cannot blow up in a finite time  $t_0$  if  $M_0, M_1, M_2 < \infty$ . Hence, we have the contradiction and the criterion (3.6) is a necessary and sufficient condition for the blow-up of local solutions of the Maxwell equation (1.2) in finite time.  $\square$

The results of Lemmas 4 and 6 can now be rewritten for the equivalent Maxwell equation (1.4) in new variables (1.3).

**Corollary 4** Fix  $s > \frac{3}{2}$  and  $C_0 > 0$  independently of  $\epsilon$ . For any  $U_0 \in H^{s+1}(\mathbb{R})$  and  $V_0 \in H^s(\mathbb{R})$  such that  $\|U_0\|_{L^\infty} \leq C_0$ , there exists an  $\epsilon$ -independent time  $T = T(\|U_0\|_{H^{s+1}} + \|V_0\|_{H^s}) > 0$  and a unique strong solution of the equivalent Maxwell equation (1.4) for any  $\epsilon \neq 0$  such that

$$U(\tau, \cdot) \in C([0, \epsilon T], H^{s+1}(\mathbb{R})) \cap C^1([0, \epsilon T], H^s(\mathbb{R})) \cap C^2([0, \epsilon T], H^{s-1}(\mathbb{R})), \quad (3.17)$$

subject to the initial data  $U(0, \cdot) = U_0$  and  $U_\tau(0, \cdot) = V_0$ . Moreover, the local solution blows up in a finite time  $\tau_0 < \infty$  if and only if

$$\limsup_{\tau \rightarrow \tau_0} (\|U(\tau, \cdot)\|_{L^\infty} + \|U_\tau(\tau, \cdot)\|_{L^\infty} + \|U_{\xi\xi}(\tau, \cdot)\|_{L^\infty}) = \infty. \quad (3.18)$$

*Proof.* The result follows from Lemmas 4 and 6 by the transformation of variables (1.3).  $\square$

To continue with the justification analysis, we decompose a solution of the Maxwell equation (1.4) in the form  $U = A + \epsilon R$ , where  $A$  is a solution of the short-pulse equation (1.1) and  $R$  is the error term satisfying

$$R_{\xi\tau} = R + \epsilon^2 R_{\tau\tau} + (3A^2 R + 3\epsilon A R^2 + \epsilon^2 R^3)_{\xi\xi} + \epsilon A_{\tau\tau}. \quad (3.19)$$

We shall now control solutions of this error equation by using apriori energy estimates.

## 4 Energy estimates for the error term

Let us define the energy for the error term,

$$E = \int_{\mathbb{R}} (R^2 + R_\xi^2 + R_{\xi\xi}^2 + 2\epsilon^2 R_\tau^2 + \epsilon^4 R_{\tau\tau}^2) dx. \quad (4.1)$$

By Sobolev embedding, the energy space  $E < \infty$  is embedded into the space of continuously differentiable functions in  $\xi$  on  $\mathbb{R}$ , which are decaying to zero at infinity as  $|\xi| \rightarrow \infty$  and are bounded by

$$\|R\|_{L^\infty} + \|R_\xi\|_{L^\infty} \leq C E^{1/2} \quad (4.2)$$

In addition, if  $A$  is a strong solution of the short-pulse equation from Lemma 1 satisfying property (2.10) of Corollary 2 and  $E < \infty$ , then  $R_{\xi\tau}$  defined by equation (3.19) belongs to  $L^2(\mathbb{R})$  such that

$$\begin{aligned} \|R_{\xi\tau}\|_{L^2} &\leq \epsilon \|A_{\tau\tau}\|_{L^2} + \|R\|_{L^2} + \epsilon^2 \|R_{\tau\tau}\|_{L^2} + 3(\|A\|_{L^\infty} + \epsilon \|R\|_{L^\infty})^2 \|R_{\xi\xi}\|_{L^2} \\ &\quad + 6\epsilon(\|A\|_{L^\infty} + \epsilon \|R\|_{L^\infty}) \|R_\xi\|_{L^2}^2 + 12(\|A\|_{L^\infty} + \epsilon \|R\|_{L^\infty}) \|A_\xi\|_{L^2} \|R_\xi\|_{L^2} \\ &\quad + 3(2\|A\|_{L^\infty} \|R\|_{L^\infty} + \epsilon \|R\|_{L^\infty}^2) \|A_{\xi\xi}\|_{L^2} + 6\|R\|_{L^\infty} \|A_\xi\|_{L^2}^2. \end{aligned}$$

The previous lengthy estimate can be greatly simplified if  $R$  belongs to the energy space (4.1) and  $A$  belongs to a ball of a finite radius  $\delta > 0$  in the function space  $C([0, T], H^s(\mathbb{R}))$  for fixed  $s > \frac{7}{2}$  and  $T > 0$ . In this case, there is an  $(\epsilon, \delta)$ -independent constant  $C > 0$  such that

$$\|R_{\xi\tau}\|_{L^2} \leq C \left( \delta\epsilon + E^{1/2} + \delta^2 E^{1/2} + \delta\epsilon E + \epsilon^2 E^{3/2} \right). \quad (4.3)$$

By Sobolev's embedding, this gives the control of  $R_\tau$  in  $L^\infty$  norm with the bound

$$\|\epsilon R_\tau\|_{L^\infty} \leq C \left( E^{1/2} + \delta\epsilon^2 + \delta\epsilon^2 E + \epsilon^3 E^{3/2} \right). \quad (4.4)$$

Moreover,  $R_\tau$  is a continuous function of  $\xi$ , which decays to zero at infinity as  $|\xi| \rightarrow \infty$ .

The main result of this section is the following lemma.

**Lemma 7** *Under the assumptions of Theorem 1, the rate of change of the energy (4.1) is given by*

$$\frac{d}{d\tau} (E + \tilde{E}) = J, \quad (4.5)$$

where

$$|\tilde{E}| \leq C \left( \epsilon E + \delta^2 E + \delta \epsilon E^{3/2} + \epsilon^2 E^2 \right), \quad (4.6)$$

$$|J| \leq C \left( \delta E^{1/2} + \delta^2 E + \delta E^{3/2} + \epsilon E^2 \right), \quad (4.7)$$

for some  $(\epsilon, \delta)$ -independent constant  $C > 0$ , as long as the solution remains in the function space

$$R \in C([0, T], H^3(\mathbb{R})) \cap C^1([0, T], H^2(\mathbb{R})) \cap C^2([0, T], H^1(\mathbb{R})). \quad (4.8)$$

The proof of Lemma 7 is based on a number of elementary but lengthy computations. Multiplying equation (3.19) by  $R_\xi$ , we derive the first balance equation,

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left( -\frac{1}{2} R_\xi^2 + \epsilon^2 R_\xi R_\tau \right) \\ & + \frac{\partial}{\partial \xi} \left( \frac{1}{2} R^2 - \frac{1}{2} \epsilon^2 R_\tau^2 + \frac{3}{2} A^2 R_\xi^2 + \frac{3}{2} (A^2)_{\xi\xi} R^2 + \epsilon A_{\xi\xi} R^3 + 3\epsilon A R R_\xi^2 + \frac{3}{2} \epsilon^2 R^2 R_\xi^2 \right) \\ & = -\epsilon R_\xi A_{\tau\tau} - 3A A_\xi R_\xi^2 + \frac{3}{2} (A^2)_{\xi\xi\xi} R^2 + \epsilon A_{\xi\xi\xi} R^3 - 9\epsilon A_\xi R R_\xi^2 - 3\epsilon A R_\xi^3 - 3\epsilon^2 R R_\xi^3. \end{aligned} \quad (4.9)$$

Multiplying equation (3.19) by  $R_\tau$ , we derive the second balance equation,

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left( \frac{1}{2} R^2 + \frac{1}{2} \epsilon^2 R_\tau^2 - \frac{3}{2} A^2 R_\xi^2 - 3\epsilon A R R_\xi^2 - \frac{3}{2} \epsilon^2 R^2 R_\xi^2 \right) \\ & + \frac{\partial}{\partial \xi} \left( -\frac{1}{2} R_\tau^2 + 3A^2 R_\xi R_\tau + 6\epsilon A R R_\xi R_\tau + 3\epsilon^2 R^2 R_\xi R_\tau \right) \\ & = -\epsilon R_\tau A_{\tau\tau} - 3A A_\tau R_\xi^2 - 3(A^2)_{\xi\xi} R R_\tau - 6A A_\xi R_\xi R_\tau \\ & \quad - 3\epsilon A_{\xi\xi} R^2 R_\tau - 6\epsilon A_\xi R R_\xi R_\tau - 3\epsilon A_\tau R R_\xi^2 - 3\epsilon A R_\xi^2 R_\tau - 3\epsilon^2 R R_\xi^2 R_\tau. \end{aligned} \quad (4.10)$$

If  $R$  belongs to the energy space  $E < \infty$ , we can integrate the balance equations (4.9) and (4.10) over  $\xi$  in  $\mathbb{R}$  and use the decay of  $R$ ,  $R_\xi$ , and  $R_\tau$  to zero at infinity as  $\xi \rightarrow \infty$ . As a result, we obtain the energy balance equation,

$$\begin{aligned} & \frac{d}{d\tau} \int_{\mathbb{R}} \left( \frac{1}{2} R^2 + \frac{1}{2} \epsilon^2 R_\tau^2 + \frac{1}{2} R_\xi^2 - \epsilon^2 R_\xi R_\tau - \frac{3}{2} A^2 R_\xi^2 - 3\epsilon A R R_\xi^2 - \frac{3}{2} \epsilon^2 R^2 R_\xi^2 \right) d\xi \\ & = \epsilon \int_{\mathbb{R}} (R_\xi - R_\tau) A_{\tau\tau} d\xi + 3 \int_{\mathbb{R}} (A A_\xi R_\xi^2 - (A A_\xi)_{\xi\xi} R^2 - A A_\tau R_\xi^2 + 2A A_\xi R R_\xi \tau) d\xi \\ & \quad + \epsilon \int_{\mathbb{R}} (-A_{\xi\xi\xi} R^3 + 9A_\xi R R_\xi^2 + 3A R_\xi^3 - 3A_{\xi\xi} R^2 R_\tau - 6A_\xi R R_\xi R_\tau - 3A_\tau R R_\xi^2 - 3A R_\xi^2 R_\tau) d\xi \\ & \quad + 3\epsilon^2 \int_{\mathbb{R}} R R_\xi^2 (R_\xi - R_\tau) d\xi, \end{aligned} \quad (4.11)$$

where the integration by parts is performed to obtain

$$\int_{\mathbb{R}} ((A^2)_{\xi\xi} RR_{\tau} + 2AA_{\xi} R_{\xi} R_{\tau}) d\xi = - \int_{\mathbb{R}} 2AA_{\xi} RR_{\tau\xi} d\xi.$$

We still need estimates of the rate of change of  $\|R_{\xi\xi}\|_{L^2}^2$  and  $\|\epsilon^2 R_{\tau\tau}\|_{L^2}^2$ . Taking the derivative of equation (3.19) in  $\xi$  and multiplying the resulting equation by  $R_{\xi\xi}$ , we derive the third balance equation,

$$\begin{aligned} & \frac{\partial}{\partial\tau} \left( -\frac{1}{2}R_{\xi\xi}^2 + \epsilon^2 R_{\xi\xi} R_{\tau\xi} \right) + \frac{\partial}{\partial\xi} \left( \frac{1}{2}R_{\xi}^2 - \frac{1}{2}\epsilon^2 R_{\tau\xi}^2 + \frac{3}{2}A^2 R_{\xi\xi}^2 + 3\epsilon AR R_{\xi\xi}^2 + \frac{3}{2}\epsilon^2 R^2 R_{\xi\xi}^2 + \frac{3}{2}\epsilon^2 R_{\xi}^4 \right) \\ = & -\epsilon R_{\xi\xi} A_{\tau\tau\xi} - 15AA_{\xi} R_{\xi\xi}^2 - 18(AA_{\xi})_{\xi} R_{\xi} R_{\xi\xi} - 6(AA_{\xi})_{\xi\xi} RR_{\xi\xi} - 3\epsilon A_{\xi\xi\xi} R^2 R_{\xi\xi} \\ & -18\epsilon A_{\xi\xi} RR_{\xi} R_{\xi\xi} - 15\epsilon A_{\xi} RR_{\xi\xi}^2 - 18\epsilon A_{\xi} R_{\xi}^2 R_{\xi\xi} - 15\epsilon AR_{\xi} R_{\xi\xi}^2 - 15\epsilon^2 RR_{\xi} R_{\xi\xi}^2. \end{aligned} \quad (4.12)$$

Finally, taking the derivative of equation (3.19) in  $\tau$  and multiplying the resulting equation by  $R_{\tau\tau}$ , we derive the last balance equation,

$$\begin{aligned} & \frac{\partial}{\partial\tau} \left( \frac{1}{2}R_{\tau}^2 + \frac{1}{2}\epsilon^2 R_{\tau\tau}^2 - \frac{3}{2}A^2 R_{\xi\tau}^2 + 3(AA_{\xi})_{\xi} R_{\tau}^2 - 3\epsilon AR R_{\xi\tau}^2 - \frac{3}{2}\epsilon^2 R^2 R_{\xi\tau}^2 \right) \\ & + \frac{\partial}{\partial\xi} \left( -\frac{1}{2}R_{\tau\tau}^2 + 3A^2 R_{\tau\tau} R_{\xi\tau} + 6\epsilon AR R_{\tau\tau} R_{\xi\tau} + 3\epsilon^2 R^2 R_{\tau\tau} R_{\xi\tau} \right) \\ = & -\epsilon R_{\tau\tau} A_{\tau\tau\tau} - 3AA_{\tau} R_{\xi\tau}^2 - 6AA_{\xi} R_{\tau\tau} R_{\xi\tau} - 6AA_{\tau} R_{\tau\tau} R_{\xi\xi} \\ & + 3(AA_{\tau})_{\xi\xi} R_{\tau}^2 - 6(AA_{\tau})_{\xi\xi} RR_{\tau\tau} - 12(AA_{\tau})_{\xi} R_{\xi} R_{\tau\tau} - 3\epsilon A_{\xi\xi\tau} R^2 R_{\tau\tau} \\ & -12\epsilon A_{\xi\tau} RR_{\xi} R_{\tau\tau} - 6\epsilon A_{\tau} (RR_{\xi})_{\xi} R_{\tau\tau} - 6\epsilon A_{\xi\xi} RR_{\tau} R_{\tau\tau} - 12\epsilon A_{\xi} R_{\xi} R_{\tau} R_{\tau\tau} \\ & -6\epsilon A_{\xi} RR_{\tau\tau} R_{\xi\tau} - 3\epsilon A_{\tau} RR_{\xi\tau}^2 - 6\epsilon AR_{\tau} R_{\tau\tau} R_{\xi\xi} - 6\epsilon AR_{\xi} R_{\tau\tau} R_{\xi\tau} - 3\epsilon AR_{\tau} R_{\xi\tau}^2 \\ & -6\epsilon^2 R_{\xi}^2 R_{\tau} R_{\tau\tau} - 6\epsilon^2 RR_{\tau} R_{\xi\xi} R_{\tau\tau} - 6\epsilon^2 RR_{\xi} R_{\xi\tau} R_{\tau\tau} - 3\epsilon^2 RR_{\tau} R_{\xi\tau}^2. \end{aligned} \quad (4.13)$$

Let us now assume the decay of  $R$ ,  $R_{\xi}$ ,  $R_{\tau}$ ,  $R_{\xi\xi}$ ,  $R_{\tau\xi}$ ,  $R_{\tau\tau}$  to zero at infinity as  $\xi \rightarrow \infty$ . The decay holds for the local solution of Corollary 4 on the short time interval  $[0, \epsilon T]$ , since the assumptions of Theorem 1 corresponds to  $s = 2 > \frac{3}{2}$  in Corollary 4. Integrating the balance equations (4.12) and (4.13) multiplied by  $\epsilon^2$  over  $\xi$  in  $\mathbb{R}$ , we obtain the energy balance equation,

$$\begin{aligned} & \frac{d}{d\tau} \int_{\mathbb{R}} \left( \frac{1}{2}\epsilon^2 R_{\tau}^2 + \frac{1}{2}\epsilon^4 R_{\tau\tau}^2 + \frac{1}{2}R_{\xi\xi}^2 - \epsilon^2 R_{\xi\xi} R_{\xi\tau} \right) d\xi \\ & + \frac{d}{d\tau} \int_{\mathbb{R}} \left( -\frac{3}{2}\epsilon^2 A^2 R_{\xi\tau}^2 + 3\epsilon^2 (AA_{\xi})_{\xi} R_{\tau}^2 - 3\epsilon^3 AR R_{\xi\tau}^2 - \frac{3}{2}\epsilon^4 R^2 R_{\xi\tau}^2 \right) d\xi \\ = & \epsilon \int_{\mathbb{R}} (R_{\xi\xi} A_{\tau\tau\xi} - \epsilon^2 R_{\tau\tau} A_{\tau\tau\tau}) d\xi + \int_{\mathbb{R}} (I_1 + \epsilon I_2 + \epsilon^2 I_3) d\xi, \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} I_1 = & 15AA_{\xi} R_{\xi\xi}^2 + 18(AA_{\xi})_{\xi} R_{\xi} R_{\xi\xi} + 6(AA_{\xi})_{\xi\xi} RR_{\xi\xi} - 3\epsilon^2 AA_{\tau} R_{\xi\tau}^2 - 6\epsilon^2 AA_{\xi} R_{\tau\tau} R_{\xi\tau} \\ & -6\epsilon^2 AA_{\tau} R_{\tau\tau} R_{\xi\xi} + 3\epsilon^2 (AA_{\tau})_{\xi\xi} R_{\tau}^2 - 6\epsilon^2 (AA_{\tau})_{\xi\xi} RR_{\tau\tau} - 12\epsilon^2 (AA_{\tau})_{\xi} R_{\xi} R_{\tau\tau} \end{aligned}$$

$$\begin{aligned}
I_2 = & 3A_{\xi\xi\xi}R^2R_{\xi\xi} + 18A_{\xi\xi}RR_{\xi}R_{\xi\xi} + 15A_{\xi}RR_{\xi\xi}^2 + 18A_{\xi}R_{\xi}^2R_{\xi\xi} + 15AR_{\xi}R_{\xi\xi}^2 \\
& - 3\epsilon^2A_{\xi\xi\tau}R^2R_{\tau\tau} - 12\epsilon^2A_{\xi\tau}RR_{\xi}R_{\tau\tau} - 6\epsilon^2A_{\tau}(RR_{\xi})_{\xi}R_{\tau\tau} - 6\epsilon^2A_{\xi\xi}RR_{\tau}R_{\tau\tau} - 12\epsilon^2A_{\xi}R_{\xi}R_{\tau}R_{\tau\tau} \\
& - 6\epsilon^2A_{\xi}RR_{\tau\tau}R_{\xi\tau} - 3\epsilon^2A_{\tau}RR_{\xi\tau}^2 - 6\epsilon^2AR_{\tau}R_{\tau\tau}R_{\xi\xi} - 6\epsilon^2AR_{\xi}R_{\tau\tau}R_{\xi\tau} - 3\epsilon^2AR_{\tau}R_{\xi\tau}^2
\end{aligned}$$

and

$$I_3 = 15RR_{\xi}R_{\xi\xi}^2 - 6\epsilon^2R_{\xi}^2R_{\tau}R_{\tau\tau} - 6\epsilon^2RR_{\tau}R_{\xi\xi}R_{\tau\tau} - 6\epsilon^2RR_{\xi}R_{\xi\tau}R_{\tau\tau} - 3\epsilon^2RR_{\tau}R_{\xi\tau}^2.$$

Recall the assumptions on  $A$  in Theorem 1. Using bounds (1.5), (4.3), and (4.4) together with the Cauchy-Schwarz inequality, we obtain (4.5), (4.6), and (4.7) from (4.11) and (4.14). The proof of Lemma 7 is complete, as long as the local solution  $R$  remain in the class of functions (4.8).

## 5 Continuation arguments and the proof of Theorem 1

We shall now finish the proof of Theorem 1. Assumption (1.5) is satisfied for a local solution of the short-pulse equation (1.1) according to Corollaries 1, 2, and 3 for any fixed  $s > \frac{7}{2}$  and  $T > 0$ . Assumptions (1.6) after the decomposition  $U = A + \epsilon R$  is rewritten in the form,

$$\|R(0, \cdot)\|_{H^2} + \|R_{\tau}(0, \cdot)\|_{H^1} \leq 1. \quad (5.1)$$

This assumption implies that the initial energy  $E|_{\tau=0} < \infty$  and  $E|_{\tau=0} = \mathcal{O}(1)$  as  $\epsilon \rightarrow 0$ , where the evolution equation (3.19) must be used. Let us denote  $E$  at the time  $\tau \geq 0$  by  $E(\tau)$ .

Since  $R(0, \cdot) \in H^3(\mathbb{R})$  and  $R_{\tau}(0, \cdot) \in H^2(\mathbb{R})$  by the assumption of Theorem 1, Corollary 4 with  $s = 2$  implies that there exists a local solution

$$R \in C([0, \epsilon T], H^3(\mathbb{R})) \cap C^1([0, \epsilon T], H^2(\mathbb{R})) \cap C^2([0, \epsilon T], H^1(\mathbb{R})) \quad (5.2)$$

of the residual equation (3.19). Because of the blow-up criterion (3.18) in Corollary 4, we can extend the existence interval to  $[0, T]$  as long as  $R$  is controlled in the energy space  $E(\tau) < \infty$  for  $\tau \in [0, T]$ .

By Lemma 7, we have

$$E(\tau) + \tilde{E}(\tau) = E(0) + \tilde{E}(0) + \int_0^{\tau} J(\tau') d\tau'. \quad (5.3)$$

We use bounds (4.6) and (4.7), the elementary bound  $2E^{1/2} \leq 1 + E$ , and Gronwall's inequality. As a result, for a sufficiently small  $\delta > 0$ , there are  $\epsilon_0 > 0$ ,  $C_0 > 0$ , and  $C_1 > 0$  such that for all  $\epsilon \in (0, \epsilon_0)$ , the following bound holds,

$$E(\tau) \leq C_0(E(0) + \delta T)e^{C_1\delta T}, \quad \tau \in [0, T]. \quad (5.4)$$

Hence, we have  $E(\tau) < \infty$  for any  $\tau \in [0, T]$ , so that the local solution of the residual equation (3.19) is extended to the whole time interval  $[0, T]$ . Because  $E(0)$  and  $T$  are  $\epsilon$ -independent, the proof of Theorem 1 is complete.

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